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Addendum

Vol. 27, No. 1 (1969), in the article, "Duffin-Kemmer Algebra as a Ring and its Representations," by Taro Shimpuku, pp. 181-207:

The present note supplements our previous article on the Duffin-Kemmer algebra, filling up gaps in the proofs of Theorems 1, 3, and 12.

DEFINITION A. For a subset I of $1 \ldots n$, $d(I)$ denotes the number of elements in I. Let $\mathscr{I}_r^n$ be the set of all subsets I of $1 \ldots n$ with $d(I) = r$ or $n - r + 1$.

LEMMA B. (1) *There exists a sign* $\alpha(I, J)$ (1 *or* -1) *for* $I, J \in \mathscr{I}_r^n$, $2r \neq n + 1$, *such that* $a_{IJ} \equiv \alpha(I, J)\, e_I \beta_{A(I,J)}$ *satisfies*

$$a_{IJ} a_{I'J'} = \delta_{JI'} a_{IJ'} \tag{A1}$$

where $A(I, J) = (k_1 k_2 \ldots)$ *is given as follows: If* $d(I) = d(J)$, *then*

$$\{k_1, k_3, \ldots\} = I \cap \bar{J},$$

$$\{k_2, k_4, \ldots\} = \bar{I} \cap J.$$

If $d(I) = n - d(J) + 1$, *then*

$$\{k_1, k_3, \ldots\} = I \cap J,$$

$$\{k_2, k_4, \ldots\} = \bar{I} \cap \bar{J}.$$

In either case, $k_1 < k_3 < \ldots$, $k_2 < k_4 < \ldots$.

(2) *For odd* n *and* $I, J \in \mathscr{I}_{(n+1)/2}^n$, *there exist signs* $\alpha^{\pm}(I, J)$ *such that*

$$a_{IJ}^{\pm} \equiv \alpha^{\pm}(I, J)\, e_I^{\pm} \beta_{A(I,J)}$$

satisfy

$$\overset{\pm}{a}_{IJ} \overset{\pm}{a}_{I'J'} = \delta_{JI'} \overset{\pm}{a}_{IJ'} \tag{A2}$$

where $A(I, J)$ *is given as in the case* $d(I) = d(J)$ *in* (1).

Proof. (1) By computations in our paper, we have

$$(e_I\beta_{A(I,J)})\,(e_{I'}\beta_{A(I',J')}) = \alpha(I, I', J')\,\delta_{I'J}(e_I\beta_{A(I,J')}),$$

where $\alpha(I, I', J') = \pm 1$. Due to the associativity:

$$(xy)\,z = x(yz),$$

$$x = e_{I_0}\beta_{A(I_0,I_1)}, \qquad y = e_{I_1}\beta_{A(I_1,I_2)}, \qquad z = e_{I_2}\beta_{A(I_2,I_3)},$$

we have

$$\alpha(I_0, I_1, I_2)\,\alpha(I_0, I_2, I_3) = \alpha(I_0, I_1, I_3)\,\alpha(I_1, I_2, I_3).$$

Hence, if we set

$$a_{IJ} = \alpha(I_0, I, J)\,e_I\beta_{A(I,J)}$$

with a fixed I_0, then it satisfies (A1).

(2) Exactly the same as (1). Q.E.D.

LEMMA C. *The two-sided ideals* $\mathfrak{A}_r$, $r = 1,\ldots, [n/2]$, *and* $\mathfrak{A}^{\pm}_{(n+1)/2}$ *are isomorphic to full matrix algebras.*

Proof. From our paper, a_{JI}, $I, J \in \mathscr{I}_r^{\,n}$, linearly span $\mathfrak{A}_r$ for $2r \neq n+1$ and $a^{\pm}_{IJ}$, $I, J \in \mathscr{I}^n_{(n+1)/2}$, linearly span $\mathfrak{A}^{\pm}_{n(+1)/2}$. Due to (A1) and (A2), the mapping from a matrix $(C_{I,J})$, $I, J \in \mathscr{I}_r^{\,n}$ to $\Sigma C_{I,J}a_{IJ}$ or $\Sigma C_{I,J}a^{\pm}_{IJ}$ is a homomorphism from a full matrix algebra onto $\mathfrak{A}_r$ or $\mathfrak{A}^{\pm}_{(n+1)/2}$. To show that it is an isomorphism, it is enough to find one nonzero element in the image.

For this purpose we consider the following representation π of $\mathfrak{B}_n$:

$$\pi(\beta_\lambda) = \tfrac{1}{2}\,(\gamma^{(1)}_\lambda \otimes 1 + 1 \otimes \gamma^{(2)}_\lambda) \tag{A3}$$

on $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ where $\gamma^{(\nu)}_\lambda$ is a representation of the Clifford algebra on $\mathfrak{H}_\nu$:

$$\gamma^{(\nu)}_\lambda\gamma^{(\nu)}_\mu + \gamma^{(\nu)}_\mu\gamma^{(\nu)}_\lambda = 2\delta_{\mu\lambda}.$$

We then have

$$\pi(e_I) = \prod_{i\in I}\{(1 + \gamma^{(1)}_i \otimes \gamma^{(2)}_i)/2\}\prod_{j\in I}\{(1 - \gamma^{(1)}_j \otimes \gamma^{(2)}_j)/2\}.$$

Hence $\operatorname{tr}\pi(e_I) = 2^{-n}\cdot\operatorname{tr} 1 \neq 0$ and hence $e_I \neq 0$. Q.E.D.

Proof of Theorem 1. The linear independence of $e_I\beta_A$ is stated in the paper, but an explicit proof has not been given there. This is now obvious from the above Lemmas. More explicitly, suppose

$$\sum_{2r\neq n+1}\sum_{I,J\in\mathscr{I}_r^n} C_{IJ}e_I\beta_{A(I,J)}+\sum_{\sigma=\pm}\sum_{I,J\in\mathscr{I}^n_{(n+1)/2}} C^\sigma_{IJ}e_I{}^\sigma\beta_{A(I,J)}=0,$$

where the second term is present only if n is odd. By inserting the left hand side between e_I and e_J and between $e_I{}^\sigma$ and $e_J{}^\sigma$, we obtain

$$C_{IJ}e_I\beta_{A(I,J)}=0,\qquad C^\sigma_{IJ}e_I{}^\sigma\beta_{A(I,J)}=0.$$

By multiplying $\beta^*_{A(I,J)}=\beta_{\underline{A}(I,J)}$, a hermite conjugate of $\beta_{A(I,J)}$, from the right, we can eliminate $\beta_{A(I,J)}$. We have already shown that $e_I\neq 0$ and hence $C_{IJ}=C^\sigma_{IJ}=0$ for all I, J, σ. Q.E.D.

Proof of Theorem 3. Let $\mathfrak{N}$ be a nilpotent ideal in $\mathfrak{B}_n$ and $m\in\mathfrak{N}$. Let $m=\sum_{I,J,\sigma}C^{(\sigma)}_{IJ}a^{(\sigma)}_{IJ}$ where (σ) indicates no superscript for $r=1,\dots,[n/2]$, $I,J\in\mathscr{I}_r{}^n$ and $\sigma=\pm$ for $I,J\in\mathscr{I}^n_{(n+1)/2}$. If one $C^{(\sigma)}_{IJ}\neq 0$, then

$$a^{(\sigma)}_{IJ}=(C^{(\sigma)}_{IJ})^{-1}\,e^{(\sigma)}_I m e^{(\sigma)}_J$$

is in $\mathfrak{N}$ because $\mathfrak{N}$ is an ideal. Hence

$$e^{(\sigma)}_I=a^{(\sigma)}_{IJ}\beta^*_{A(I,J)}\in\mathfrak{N}.$$

However $(e^{(\sigma)}_I)^2=e^{(\sigma)}_I\neq 0$, which is in contradiction with the assumption that $\mathfrak{N}$ is a nilpotent ideal. Hence $m=0$ and $\mathfrak{B}_n$ does not have a nontrivial nilpotent ideal. Q.E.D.

Proof of Theorem 12. Let $a=\sum_{r,\sigma}a^{(\sigma)}_r$, $a^{(\sigma)}_r\in\mathfrak{A}^{(\sigma)}_r$ where (σ) indicates no superscript for $r=1,\dots,[n/2]$ and $\sigma=\pm$ for $r=(n+1)/2$, if n is odd. Since a full matrix algebra is regular, there exists $b^{(\sigma)}_r\in\mathfrak{A}^{(\sigma)}_r$ such that $a^{(\sigma)}_r b^{(\sigma)}_r a^{(\sigma)}_r=a^{(\sigma)}_r$. Then $b=\sum_{r,\sigma}b^{(\sigma)}_r$ satisfies $aba=a$. Q.E.D.

Theorem 11 is false for a general n. Let $a=a_1+a_3$, $b=b_2+b_3$, $a_1\in\mathfrak{A}_1$, $b_2\in\mathfrak{A}_2$, a_3, $b_3\in\mathfrak{A}_3$. Then $a\mathfrak{B}_n b=\mathfrak{A}_3$ but $a\notin\mathfrak{A}_3$, $b\notin\mathfrak{A}_3$.

Finally, a mistake in writing in the paper should be corrected as follows: In the last line of p. 202, $\binom{n+1}{s}$, the order of matrix of right ideals $\mathfrak{r}^+_{i_1\cdots i_s}$, $\mathfrak{r}^-_{i_1\cdots i_s}$ should be $\frac{1}{2}\binom{n+1}{s}$. On p. 206, line 16, $\beta_{\underline{X}}\beta_X=e_J$ and $\beta_X\beta_{\underline{X}}=e_I$ should be read $e_J\beta_{\underline{X}}\beta_X e_J=e_J$ and $e_I\beta_X\beta_{\underline{X}}e_I=e_I$. Similarly on p. 206, line 2 from the bottom in text, $\beta_C\beta_{\underline{C}}=e_I$ and $\beta_{\underline{C}}\beta_C=e_J$ should be read $e_I\beta_C\beta_{\underline{C}}e_I=e_I$ and $e_J\beta_{\underline{C}}\beta_C e_J=e_J$.

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